STABILITY OF MOTION IN A CRITICAL CASE

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We consider a stability of motion defined by a system of differential equations of perturbed motion of the type

$$x_i^{\cdot} = y_i + X_i^*, \quad y_i^{\cdot} = Y_i^*, \quad \zeta_s^{\cdot} = \sum_{\kappa=1}^{n} p_{sk}\,\xi_k + Z_s^* \qquad \left(\begin{matrix} i = 1, \ldots, m \\ s = 1, \ldots, n \end{matrix}\right) \qquad (0.1)$$

where X_i^*, Y_i^* and Z_s^* are holomorphic functions containing no terms of order lower than second in $x_1, \ldots, x_m$; $y_1, \ldots, y_m$; $s_1, \ldots, s_n$. All roots of Eq. $|p_{sk} - \delta_{sk}\lambda| = 0$ have, different from zero, negative real parts.

Let us consider a system (0.1) with conditions

$$Y_i^* = \sum_{k=1}^{m} a_{ik} y_k^2 + \sum_{k=1}^{m} P_{ik}(\zeta_1, \ldots, \zeta_n)\, y_k + Q_i(\zeta_1, \ldots \zeta_n) +$$

$$+ \sum_{\sigma=1}^{n} \zeta_\sigma \varphi_{i\sigma}(x_1, \ldots, x_m) + R_i(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ \zeta_1, \ldots, \zeta_n) \qquad (0.2)$$

$$Z_s^* = \sum_{\sigma=1}^{n} \zeta_\sigma \omega_{s\sigma}(x_1, \ldots, x_m) + R_s'(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ \zeta_1, \ldots, \zeta_n)$$

where a_{ik} are constants; $\varphi_{i\sigma}$ and $\omega_{s\sigma}$ are holomorphic functions which vanish when $x_1 = \ldots = x_m = 0$; P_{ik} are linear and Q_i are quadratic forms in $s_1, \ldots, s$; R_i and R_s' are holomorphic functions in $x_1, \ldots, x_m$; $y_1, \ldots, y_m$; and $\zeta_1, \ldots, \zeta_n$, containing no terms of order lower than third in these variables. Stability of this system was investigated in [1]. We attempt to show that the unperturbed motion is unstable when X_i^* and Z_s^* satisfy the conditions (0.2).

Although the functions Y_i^* and Z_s^* investigated in [1] represent a particular form throughout, function V proposed by the authors of [1] is not a Chetaev function unless additional conditions are imposed on Y_i^* and Z_s^*. Indeed, expression ([1], (2.8)) e.g. contains aggregates of the form

$$\sum_{k=1}^{m} \left[1 + \left(1 - \sum_{k=1}^{m} a_{ik}\right) x_k\right] R_i, \qquad \sum_{k=1}^{m} \sum_{s=1}^{n} \psi_{ks}(x_1, \ldots, x_m)\, R_s'$$

which include terms such as

$$x_i^{\delta_i}, \quad y_i x_i^{\nu_i}, \quad \zeta_s x_i^{\mu_i} \quad (\delta_i,\ \mu_i \nu_i \geqslant 2)$$

Obviously, in presence of such terms, dV/dt can assume, for $V > 0$, values of any sign. It follows therefore that additional conditions must be imposed on Y_i^* and Z_s^* when choosing V according to [1], (2.5). These conditions are:

1) When $y_1 = \ldots = y_m = \zeta_1 = \ldots = \zeta_n = 0$ then all $Y_i^* \equiv 0$, and all $Z_s^* \equiv 0$.

2) All R_i and R_s' do not contain terms of order lower than second in $y_1, \ldots, y_m$ and $s_1, \ldots, s_n$.

1. Consider a system of Eqs. (0.1) assuming that X_i^* and Z_s^* vanish when $y_1 = \ldots = y_m = s_1 = \ldots = s_n = 0$. This assumption does not reduce the generality of our problem [1].

Let us transform (0.1), putting

$$\zeta_s = z_s + u_s(x_1, \ldots, x_m;\ y_1, \ldots, y_m) \qquad (s = 1, \ldots, n) \tag{1.1}$$

where $u_s\,(x_1,\ldots,\ x_m;\ y_1,\ldots,\ y_m)$ are roots of

$$p_{s1}u_1 + \cdots + p_{sn}u_n + Z_s^*(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ u_1, \ldots, u_n) = 0$$

As a result we obtain

$$x_s^{\cdot} = y_i + X_i, \quad y_i^{\cdot} = Y_i \quad z_s^{\cdot} = \sum_{k=1}^{n} p_{sk}z_k + Z_s \quad \begin{pmatrix} i = 1, \ldots, m \\ s = 1, \ldots, n \end{pmatrix} \tag{1.2}$$

where X_i and Y_i are the values of functions X_i^* and Y_i^* when

$$\zeta_s = z_s + u_s, \text{ a} \tag{1.3}$$

$$z_s = z_s^*(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ z_1 + u_1, \ldots, z_n + u_n) -$$

$$- z_s^*(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ u_1, \ldots, u_n) - \sum_{k=1}^{m} (y_k + X_k)\frac{\partial u_s}{\partial x_k} - \sum_{k=1}^{m} Y_k \frac{\partial u_s}{\partial y_k}$$

We note that

a) when $y_1 = \ldots = y_m = z_1 = \ldots = z_m = 0$ then $X_i \equiv 0$.

b) functions Y_i and Z_s will also vanish identically when $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$, provided that all Y_i^* become identically zero when $y_1 = \ldots = y_m = s_1 = \ldots = s_n = 0$.

c) if all Y_i vanish when $y_1 = \ldots = y_m = s_1 = \ldots = s_n = 0$ and Y_i does not contain terms of first order in $y_1,\ldots,\ y_m$ with $z_1 = \ldots = z_n = 0$, then functions Z_s will not contain terms of first order in $y_1,\ldots,\ y_m$ when $z_1 = \ldots = z_n = 0$.

Let us now assume that $Y_i = 0\ (i = 1,\ldots,m)$ with $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$ and, that when $z_1 = \ldots = z_n = 0$, then functions Y_i do not contain linear terms in $y_1,\ldots,y_m$. We shall show that the unperturbed motion is unstable.

Let us take the Chetaev function in the form

$$V = \sum_{i=1}^{m} x_i y_i + \sum_{s=1}^{n} z_s v_s(x_1, \ldots, x_m) + W(z_1, \ldots, z_n) \tag{1.4}$$

where $W(z_1,\ldots,\ z_n)$ is a negative definite quadratic form satisfying Eq.

$$\sum_{s=1}^{n} \frac{\partial W}{\partial z_s}(p_{s1}z_1 + \cdots + p_{sn}z_n) = \sum_{s=1}^{n} z_s^2$$

and v_s are holomorphic functions of $x_1,\ldots,x_m$ which become zero when $x_1 = \ldots = x_m = 0$ and which satisfy Eqs.

$$\sum_{i=1}^{m} x_i F_{ik} + \sum_{s=1}^{n} v_s(p_{sk} + Q_{sk}) = 0, \qquad (k = 1, \ldots, n)$$

in which F_{ik} and Q_{sk} are

$$F_{ik} = \left.\frac{\partial Y_i}{\partial z_k}\right|_{z=y=0}, \qquad Q_{sk} = \left.\frac{\partial z_s}{\partial z_k}\right|_{z=y=0}$$

When functions v_s are chosen in this manner, then the derivative V' can, by virtue of the structure of right-hand side parts of the system (1.2), be represented by

$$V' = \sum_{i=1}^{m} y_i^2 + \sum_{s=1}^{n} z_s^2 + \sum_{i=1}^{m}\sum_{k=1}^{m} y_i y_k \varphi_{ik} + \sum_{j=1}^{n}\sum_{\sigma=1}^{n} z_j z_\sigma \psi_{j\sigma} + \sum_{i=1}^{m}\sum_{j=1}^{n} y_i z_j f_{ij}$$

where φ_{ik}, and $\Psi_{j\sigma}$ f_{ij} become zero when

$$x_1 = \cdots = x_m = y_1 = \cdots = y_m = z_1 = \cdots = z_n = 0.$$

Let us consider the region $V > 0$. Obviously within this region V' is positive and becomes zero only on the boundary of the region $V > 0$ where $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$. Hence the unperturbed motion is unstable [2].

Note. It can easily be shown that in this case motions $x_i = c_i (i = 1,\ldots, m)$, $y_1 = \ldots = y_n = z_1 = \ldots = z_n = 0$ will, for sufficiently small c_i, also be unstable.

2. We shall now consider the case when $Y_i \not\equiv 0$ while $y_1 = \ldots = y_m = z_1 = \ldots = z_n = 0$. Assume that when $z_1 = \ldots = z_n = 0$ Y_i has no terms linear in $y_1, \ldots y_m$ and that in (1.2) we have

$$Y_i = \sum_{k=1}^{m} g_{ik} x_k^{r_{ik}} + \sum_{k=1}^{m} x_k^{r_{ik}} f_{ik}(x_1, \ldots, x_m) + \sum_{k=1}^{m} a_{ik} y_k^2 +$$
$$+ \sum_{\sigma=1}^{n} z_\sigma \varphi_{i\sigma}(x_1, \ldots, x_m) + \sum_{k=1}^{m} P_{ik}(z_1, \ldots, z_n) y_k + Q_i(z_1, \ldots, z_n) +$$
$$+ R_i(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ z_1, \ldots, z_n) \tag{2.1}$$
$$Z_s = \sum_{\sigma=1}^{n} z_\sigma \omega_{s\sigma}(x_1, \ldots, x_n) + R_s'(x_1, \ldots, x_m;\ y_1, \ldots, y_m;\ z_1, \ldots, z_n)$$
$$(i = 1, \ldots, m;\ s = 1, \ldots, n)$$

where a_{ik} and q_{ik} are constants; f_{ik}, $\varphi_{i\sigma}$ and $\omega_{s\sigma}$ are holomorphic functions of $x_1, \ldots, x_m$ vanishing when $x_1 = \ldots = x_m = 0$; P_{ik} are linear and Q_i are quadratic forms in $z_1, \ldots, z_n$; R_i are holomorphic functions of $x_1, \ldots, x_m$; $y_1, \ldots, y_m$; and $z_1, \ldots, z_m$, containing no terms of order lower than third in the above variables and no terms of order lower than second in $y_1 \ldots y_m$ and $z_1, \ldots, z_n$; R_s' are holomorphic functions of $x_1, \ldots, x_m$; $y_1, \ldots, y_m$ and $z_1, \ldots, z_n$, vanishing when $x_k = y_k = z_s = 0$, and containing no terms linear in $z_1, \ldots, z_n$ when $y_1 = \ldots = y_m = 0$. We should note that if functions Y_i contain e.g. terms $x_i^{r_{ik}}$ when $z_1 = \ldots = z_n = 0$ then functions z_s may contain analogous terms when $z_1 = \ldots = z_n = 0$ and their order in x_i will be, at most, higher by one. This property of Z_s follows from (1.3).

We shall now show that the unperturbed motion is unstable, if:

a) nonlinear functions Y_i and Z_s satisfy conditions (2.1)

b) in each column of the matrix

$$\left\| \begin{matrix} r_{11} & r_{12} & \ldots & r_{1m} \\ \cdot & \cdot & \cdot & \cdot \\ r_{m1} & r_{m2} & \ldots & r_{mm} \end{matrix} \right\| \tag{2.2}$$

the smallest numbers $r_{i'k}(k = 1, \ldots m)$ are even and corresponding magnitudes $g_{i'k}$ are of the same sign.

Let us take the Liapunov function V of the form [1]

$$V = \sum_{k=1}^{m} \left[1 + \left(\sum_{i'} g_{i'k} - \sum_{i=1}^{m} a_{ik}\right) x_k\right] y_k + \sum_{k=1}^{m} \sum_{s=1}^{n} z_s \psi_{ks} +$$
$$+ \sum_{i=1}^{m} \sum_{k=1}^{m} U_{ik}(z_1, \ldots, z_n) y_k + \sum_{k=1}^{m} W_k \tag{2.3}$$

where ψ_{ks} are functions of $x_1, \ldots, x_m$ which satisfy Eqs.

$$\sum_{s=1}^{n} (p_{s\sigma} + \omega_{s\sigma}) \psi_{ks} + \left[1 + \left(\sum_{i'} g_{i'k} - \sum_{i=1}^{m} a_{ik}\right) x_k\right] \varphi_{k\sigma} = 0$$
$$(\sigma = 1, \ldots, n;\ k = 1, \ldots, m)$$

U_{ik} and W_k are linear and quadratic forms of $z_1, \ldots, z_n$ defined from Eqs.

$$\sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial z_s} (p_{s1} z_1 + \cdots + p_{sn} z_n) + P_{ik} = - \sum_{s=1}^{n} z_s \left(\frac{\partial \psi_{is}}{\partial x_k}\right)_0$$
$$\sum_{s=1}^{n} \frac{\partial W_k}{\partial z_s} (p_{s1} z_1 + \cdots + p_{sn} z_n) + Q_k = \sum_{i'} \sum_{s=1}^{n} g_{i'k} z_s^2$$

Function V of the form (2.3) satisfies the Liapunov theorem on instability [3], therefore the unperturbed motion is unstable. It is also unstable if:

a) nonlinear functions Y_i and Z_s satisfy the conditions (2.1);

b) diagonal elements r_{kk} of the matrix (2.2) are odd and smaller than the elements of the corresponding column, and $g_{kk} > 0$.

In this case the Liapunov function V can be written as

$$V = \sum_{k=1}^{m} \left(g_{kk} x_k + \sum_{i=1}^{m} U_{ik}\right) y_k + \sum_{k=1}^{m} \sum_{s=1}^{n} z_s \psi_{ks} + W \tag{2.4}$$

where ψ_{ks} are functions of $x_1, \ldots, x_m$, satisfying

$$\sum_{s=1}^{n} (p_{s\sigma} + \omega_{s\sigma}) \psi_{ks} + g_{kk} x_k \varphi_{k\sigma} = 0 \qquad \begin{pmatrix} \sigma = 1, \ldots, n \\ k = 1, \ldots, m \end{pmatrix}$$

Functions U_{ik} and W are linear and quadratic forms given by

$$\sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial z_s} (p_{s1} z_1 + \cdots + p_{sn} z_n) = - \sum_{s=1}^{n} z_s \left(\frac{\partial \psi_{is}}{\partial x_k}\right)_0 \qquad \begin{pmatrix} i = 1, \ldots, m \\ k = 1, \ldots, m \end{pmatrix}$$

$$\sum_{s=1}^{n} \frac{\partial W}{\partial z_s} (p_{s1} z_1 + \cdots + p_{sn} z_n) = \sum_{s=1}^{n} z_s^2$$

Function V of the form (2.4) satisfies the Liapunov theorem on instability [3], therefore the unperturbed motion is unstable.

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